Hand-In Exercise 1: Modelling

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1 System Modeling

This section contains a derivation of the equations of motion for a robot arm consisting of the three first links of a UR5e robot. The model is derived using Lagrange–D'Alembert's Principle.

An illustration of the robot is shown in Figure 1 that includes the kinematic parameters specified in Table 1. The figure also show position vectors for the center of mass of each joint; the coordinates of the center of masses are specified in (1).

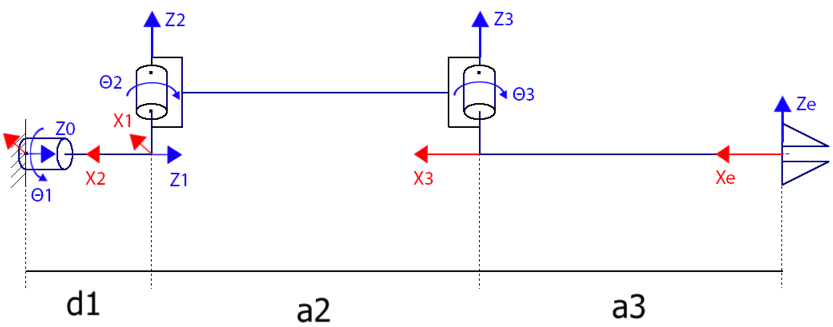

Figure 1: Illustration of considered robot, including kinematic parameters and reference frames.

Link	a_i	α_i	d_i	θ_i
1	0	$\frac{\pi}{2}$	d_1	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Table 1: DH-parameters for the 3-link robot arm.

$$\boldsymbol{p}_{l_1}^1 = \begin{bmatrix} p_{l_1,x}^1 \\ p_{l_1,y}^1 \\ p_{l_1,z}^1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.02561 \\ 0.00193 \end{bmatrix}, \quad \boldsymbol{p}_{l_2}^2 = \begin{bmatrix} p_{l_2,x}^2 \\ p_{l_2,y}^2 \\ p_{l_2,z}^2 \end{bmatrix} = \begin{bmatrix} 0.2125 \\ 0 \\ 0.11336 \end{bmatrix}, \quad \boldsymbol{p}_{l_3}^3 = \begin{bmatrix} p_{l_3,x}^3 \\ p_{l_3,y}^3 \\ p_{l_3,z}^3 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.0 \\ 0.0265 \end{bmatrix} \tag{1}$$

To derive a dynamical model of the robot, the mass and inertia tensor of each link are given in

$$I_1^1 = \begin{bmatrix} 0.0084 & 0 & 0 \\ 0 & 0.0064 & 0 \\ 0 & 0 & 0.0084 \end{bmatrix}, I_2^2 = \begin{bmatrix} 0.0078 & 0 & 0 \\ 0 & 0.21 & 0 \\ 0 & 0 & 0.21 \end{bmatrix}, I_3^3 = \begin{bmatrix} 0.0016 & 0 & 0 \\ 0 & 0.0462 & 0 \\ 0 & 0 & 0.0462 \end{bmatrix} \tag{2}$$

Finally, the masses of the links are $m_1 = 3.761$ kg, $m_2 = 8.058$ kg, $m_3 = 2.846$ kg.

1.1 Robot Kinematics

The kinematics of the robot are given by the homogeneous transformations

$$A_1^0 = \begin{bmatrix} c_1 & -s_1\cos(\alpha_1) & s_1\sin(\alpha_1) & a_1c_1 \\ s_1 & c_1\cos(\alpha_1) & -c_1\sin(\alpha_1) & a_1s_1 \\ 0 & \sin(\alpha_1) & \cos(\alpha_1) & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

$$A_2^1 = \begin{bmatrix} c_2 & -s_2\cos(\alpha_2) & s_2\sin(\alpha_2) & a_2c_2 \\ s_2 & c_2\cos(\alpha_2) & -c_2\sin(\alpha_2) & a_2s_2 \\ 0 & \sin(\alpha_2) & \cos(\alpha_2) & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

$$A_3^2 = \begin{bmatrix} c_3 & -s_3\cos(\alpha_3) & s_3\sin(\alpha_3) & a_3c_3 \\ s_3 & c_3\cos(\alpha_3) & -c_3\sin(\alpha_3) & a_3s_3 \\ 0 & \sin(\alpha_3) & \cos(\alpha_3) & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$ in addition to the transformations

$$T_2^0 = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & a_2c_1c_2 \\ c_2s_1 & -s_1s_2 & -c_1 & a_2c_2s_1 \\ s_2 & c_2 & 0 & d_1 + a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

$$T_3^0 = \begin{bmatrix} c_{23}c_1 & -s_{23}c_1 & s_1 & c_1(a_3c_{23} + a_2c_2) \\ c_{23}s_1 & -s_{23}s_1 & -c_1 & s_1(a_3c_{23} + a_2c_2) \\ s_{23} & c_{23} & 0 & d_1 + a_3s_{23} + a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

where $c_{ij} = \cos(\theta_i + \theta_j)$ and $s_{ij} = \sin(\theta_i + \theta_j)$.

1.2 Center of Mass of Links

The center of mass for each link have the following coordinates

$$p_{c1}^0 = \begin{bmatrix} p_{l_1,x}^1 c_1 + p_{l_1,z}^1 s_1 \\ p_{l_1,x}^1 s_1 - p_{l_1,z}^1 c_1 \\ p_{l_1,y}^1 + d_1 \\ 1 \end{bmatrix} \tag{8}$$

$$p_{c2}^0 = \begin{bmatrix} c_1c_2p_{l_2,x}^2 - c_1s_2p_{l_2,y}^2 + s_1p_{l_2,z}^2 + a_2c_1c_2 \\ c_2s_1p_{l_2,x}^2 - s_1s_2p_{l_2,y}^2 - c_1p_{l_2,z}^2 + a_2c_2s_1 \\ s_2p_{l_2,x}^2 + c_2p_{l_2,y}^2 + d_1 + a_2s_2 \\ 1 \end{bmatrix} \tag{9}$$

$$p_{c3}^0 = \begin{bmatrix} c_{23}c_1p_{l_3,x}^3 - s_{23}c_1p_{l_3,y}^3 + s_1p_{l_3,z}^3 + c_1\left(a_3c_{23} + a_2c_2\right) \\ c_{23}s_1p_{l_3,x}^3 - s_{23}s_1p_{l_3,y}^3 - c_1p_{l_3,z}^3 + s_1\left(a_3c_{23} + a_2c_2\right) \\ s_{23}p_{l_3,x}^3 + c_{23}p_{l_3,y}^3 + d_1 + a_3s_{23} + a_2s_2 \\ 1 \end{bmatrix} \tag{10}$$

where $c_{ij} = \cos(\theta_i + \theta_j)$ and $s_{ij} = \sin(\theta_i + \theta_j)$.

1.3 Verification of Symbolic Model

Term	`SolutionID`
$g(q)$	92b40e6a-b2f8-4648-85d6-742989f3dd16
$B(q)$	7c04582f-c1db-4c2b-9c17-f8f37e520e87
$C(q, \dot{q})$	12e444bd-9e30-4aa1-957d-f1ce3e90f931
$J_e(q)$	0f7be2d6-caf0-4600-8fa2-67d284696a7f

Table 2: Please insert solution ID for each term. The Jacobian J_e is with respect to the end-effector.

1.4 Jacobian of the Tool Frame

The robot has a tool mounted to the end-effector with a transformation given by T_{tip}^{TCP}. If the force can be measured at the tooltip, the robot dynamical model changes to

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau - J^T(q)h_e, \tag{11}$$

where B(q) is the inertia tensor, C(q,) is a matrix containing Coriolis and centrifugal terms, g(q) is the gravity vector and τ is the actuator torque.

$$J = \begin{bmatrix} c_1 p_{l3x}^3 - s_1(a_2c_2 + a_3c_{23}) - c_{23}p_{l3x}^3 + p_{l3y}^3 s_{23} + 0,3 & 0,3 - c_1\sigma_1 & 0,3 - c_1\sigma_2 \\ p_{l3x}^3 s_1 + c_1(a_2c_2 + a_3c_{23}) + c_1c_{23}p_{l3x}^3 - c_1p_{l3y}^3 s_{23} & -s_1\sigma_1 & -s_1\sigma_2 \\ 0,05 & a_2c_2 + a_3c_{23} + c_{23}p_{l3x}^3 - p_{l3y}^3 s_{23} + 0,05 & a_3c_{23} + c_{23}p_{l3x}^3 - p_{l3y}^3 s_{23} + 0,05 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where $\sigma_1 = c_{23}p_{l3y}^3 + a_2s_2 + a_3s_{23} + p_{l3x}^3 s_{23}$
$\sigma_2 = c_{23}p_{l3y}^3 + a_3s_{23} + p_{l3x}^3 s_{23}$
$c_{ij} = \cos(\theta_i + \theta_j)$ and $s_{ij} = \sin(\theta_i + \theta_j)$.

1.5 Simulation of Model

Simulate the robot arm with input $\boldsymbol{Q} = (\tau_1, \tau_2, \tau_3) = -D\dot{\boldsymbol{q}}$, where $D = 5I_3$. Use initial condition $\boldsymbol{q} = (\theta_1, \theta_2, \theta_3) = (1, \pi/3, \pi/3)$.
A tool is mounted to the robot end effector with a translation matrix defined as

$$T_{\text{tip}}^{TCP} = \begin{bmatrix} 1 & 0 & 0 & 0.3 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0.05 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

which is illustrated on Figure 2. The following force profile is measured at the tool tip,

$$h_{\text{tip}}(t) = \begin{cases} 0 & \text{if } t \leq 5\,\text{s} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}^\intercal & \text{if } 5\,\text{s} < t \leq 10\,\text{s} \\ 0 & \text{if } t > 10s \end{cases}. \tag{13}$$

Simulate the system for 20 s, and plot the results in Figure 3.

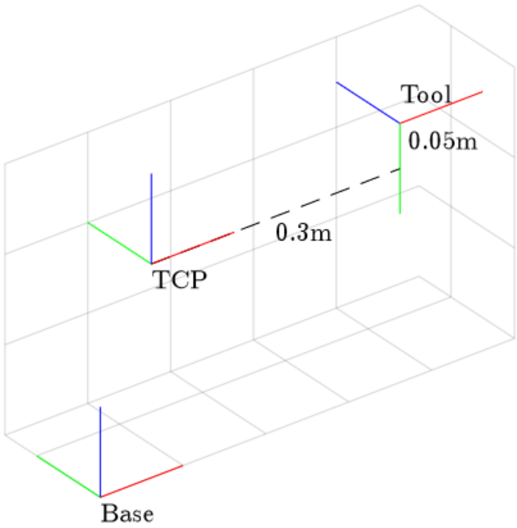

Figure 2: Illustration of the base, TCP and tip frame.

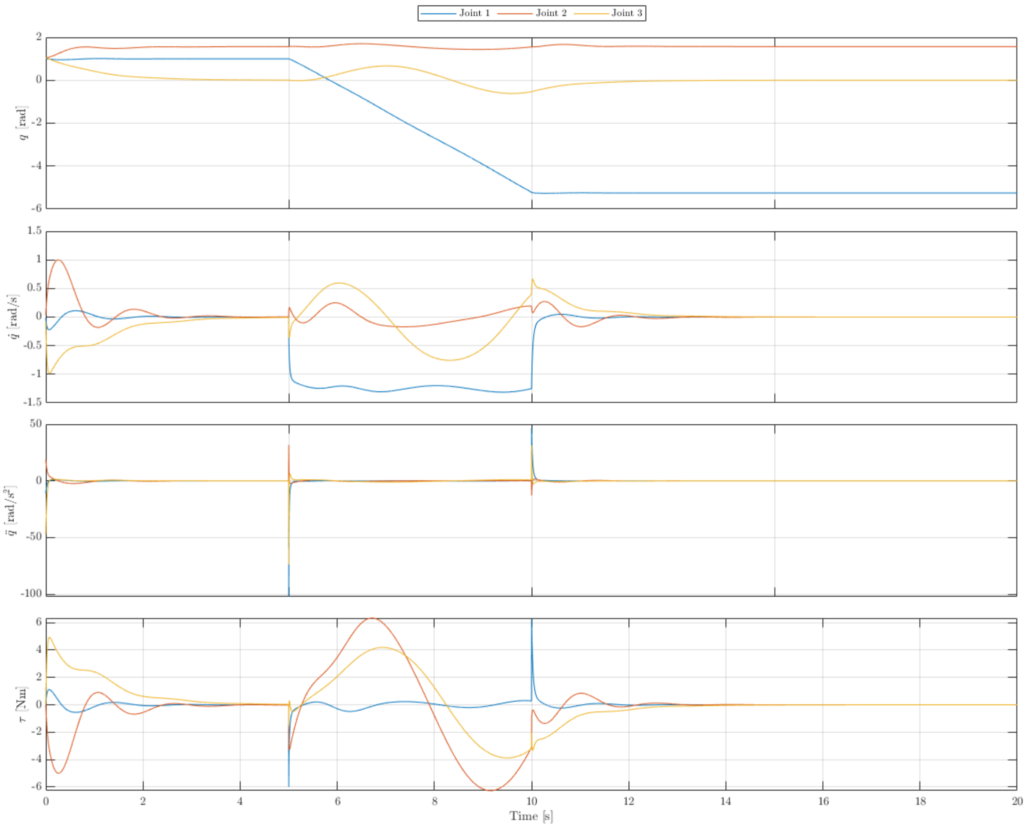

Figure 3: Simulation Results.